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Geometric Tomography

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0

Background material

This chapter introduces notation and terminology and summarizes aspects of the theories of affine and projective transformations, convex and star sets, and measure and integration appearing frequently in the sequel.

Some passages are designed to ease the beginner into these areas, but not all the material is elementary. It is intended that the reader **start with Chapter 1**, and use the present chapter as a reference manual. For Chapter 1, *the requisite material is included in the first four sections of this chapter only*, and for Chapter 2, *the requisite material is included in the first five sections only.*

0.1. Basic concepts and terminology

This section is a brief review of some basic definitions and notation. Any unexplained notation can be found in the list at the end of the book.

Almost all the results in this book concern Euclidean n-dimensional space $\mathbb{E}^n$. The origin in $\mathbb{E}^n$ is denoted by o, and if $x \in \mathbb{E}^n$, we usually label its coordinates by $x = (x_1, \ldots, x_n)$. (In $\mathbb{E}^2$ and $\mathbb{E}^3$ we often use a different letter for a point and label its coordinates in the traditional way by x, y, and z.) The Euclidean norm of x is denoted by $\|x\|$, and the Euclidean scalar product of x and y by $x \cdot y$. The closed line segment joining x and y is $[x, y]$. Points are identified with vectors, and are always denoted by lowercase letters. For sets we usually employ capitals, although we also use lowercase for straight lines. Script capitals are used for classes of sets; an exception is the $\mathcal{S}$ we use for sets of directions in Chapters 1 and 2, but here we are really identifying a direction with the line through the origin parallel to it. The natural numbers, real numbers, and complex numbers have the usual symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$. The letters i, j, k, m, and n denote integers unless it is stated otherwise (in parts of the book i often represents a real number), or unless we are working with complex numbers, when $i^2 = -1$

as usual. In particular, the default meaning of an expression such as $1 \leq i \leq n$ is $i \in \{1, \ldots, n\}$.

The *unit ball* in $\mathbb{E}^n$ is $B = \{x : \|x\| \leq 1\}$, with surface the *unit n-sphere* $S^{n-1} = \{x : \|x\| = 1\}$. When necessary we may write B^n instead of B. We attempt to reserve u for the members of S^{n-1}, the unit vectors. If $u \in S^{n-1}$, then $u^\perp$ is the $(n-1)$-dimensional subspace orthogonal to u, and l_u the 1-dimensional subspace parallel to u. Generally, S is used for a subspace, and $S^\perp$ for its complementary orthogonal subspace. The Grassmann manifold of k-dimensional subspaces of $\mathbb{E}^n$ is denoted by $\mathcal{G}(n, k)$. More often than not the topology on $\mathcal{G}(n, k)$ is unnecessary, and the symbol then simply denotes the corresponding set of subspaces.

Translates of subspaces are called *planes* or *flats*, or *hyperplanes* if they are $(n-1)$-dimensional. A hyperplane divides the space into two *half-spaces* (*half-planes* in $\mathbb{E}^2$). A *ray* is a semi-infinite straight line. If E is a set, the *linear hull* $\operatorname{lin} E$ and *affine hull* $\operatorname{aff} E$ of E are, respectively, the smallest subspace and the smallest plane containing E. The *dimension* $\dim E$ of a set E is the dimension of its affine hull.

We say that two planes are *parallel* if one is contained in a translate of the other, and *orthogonal* if, when translated so that they contain the origin, one contains the complementary orthogonal subspace of the other. (These terms are often used by other authors in a more restrictive way.) A *slab* is the closed region between two parallel hyperplanes.

Suppose that F_1, F_2 are planes in $\mathbb{E}^n$, of dimensions d_1 and d_2, respectively. Then by [45, Theorem 32.1], either $F_1 \cap F_2 = \emptyset$ or $\dim(F_1 \cap F_2) \geq d_1 + d_2 - n$. The planes F_1 and F_2 are in *general position* with respect to each other if either $d_1 + d_2 < n$, $F_1 \cap F_2 = \emptyset$, and there is no direction parallel to both planes, or $d_1 + d_2 \geq n$ and $\dim(F_1 \cap F_2) = d_1 + d_2 - n$. See [45, pp. 88–90] for more information. A finite set of points in $\mathbb{E}^n$ is said to be in *general position* if no more than $k + 1$ of them belong to any k-dimensional plane.

A few of our results are set in 2-dimensional projective space $\mathbb{P}^2$. Generally, *n-dimensional projective space* $\mathbb{P}^n$ can be defined as the space of 1-dimensional subspaces of $\mathbb{E}^{n+1}$. The points of $\mathbb{P}^n$ are labeled by *homogeneous coordinates* $w = (w_1, \ldots, w_{n+1})$, not all zero, so for real $t \neq 0$ the points w and tw are identified; see, for example, [45, p. 217]. In this way, $\mathbb{P}^1$ can be regarded as the unit circle S^1 with antipodal points identified. We can also identify $\mathbb{E}^n$ with $\{w : w_{n+1} \neq 0\}$, where the usual coordinates are given by $x_i = w_i / w_{n+1}$. The remaining set $H_\infty = \{w : w_{n+1} = 0\}$ is the *hyperplane at infinity* (strictly speaking, a copy of $\mathbb{P}^{n-1}$). In particular, $\mathbb{P}^2$ can be regarded as $\mathbb{E}^2$ with a *line at infinity* (strictly speaking, a copy of $\mathbb{P}^1$) adjoined.

Our terminology for set theory and topology is standard. If E is a set, then $|E|$, $\operatorname{co} E$, $\operatorname{cl} E$, $\operatorname{int} E$, and $\operatorname{bd} E$ denote the *cardinality*, *complement*, *closure*, *interior*, and *boundary* of E, respectively; also, $\operatorname{relint} E$ is the *relative interior* of E, that is, the interior of E relative to $\operatorname{aff} E$. The *relative boundary* of E is the boundary of

E relative to aff E. The *symmetric difference* of E and F is

$$E \triangle F = (E \setminus F) \cup (F \setminus E).$$

A G_δ *set* is a countable intersection of open sets, and an F_σ *set* is a countable union of closed sets. A set is of *first category* if it is the countable union of nowhere dense sets, and of *second category* otherwise. A set in a complete metric space is of second category if and only if it is the complement of a set of first category, and this occurs if and only if it contains a dense G_δ set; see, for example, [437, pp. 158–60]. A *component* of a set is a maximal connected subset. A closed set is *regular* if it is the closure of its interior, and a *body* is a compact, regular set.

The *diameter* diam E of a set E is

$$\operatorname{diam} E = \sup\{\|x - y\| : x, y \in E\}.$$

If x is a point and E is a closed set, the *distance* between x and E is

$$d(x, E) = \inf\{\|x - y\| : y \in E\}.$$

If E and F are sets, and r is a real number, then

$$E + F = \{x + y : x \in E, y \in F\},$$

and

$$rE = \{rx : x \in E\}.$$

A set E is called *centered* if $-x \in E$ whenever $x \in E$, and *centrally symmetric* if there is a vector c such that the translate $E - c$ of E by $-c$ is centered. In the latter case c is called a *center* of E. The center of a nonempty bounded centrally symmetric set is unique.

If X is a subset of $\mathbb{E}^n$, or indeed any topological space, the *support* of a real-valued function f on X is the set cl $\{x \in X : f(x) \neq 0\}$. We denote by $C(X)$ the class of continuous real-valued functions on X. When X is an appropriate subset of $\mathbb{E}^n$, $C_e(X)$ denotes the even functions in $C(X)$, and $C_e^+(X)$ the nonnegative functions in $C_e(X)$.

If f and g are real-valued functions, we say that $f = O(g)$ on $A \subset \mathbb{R}$ if there is a constant c such that $|f(x)| \leq c|g(x)|$ for all $x \in A$. When $A = \mathbb{N}$, we sometimes say that $f = O(g)$ as $n \to \infty$, while $f = O(g)$ as $x \to 0$ means that $f = O(g)$ on $A = (0, a)$ for sufficiently small a.

0.2. Transformations

No single book seems to provide a completely satisfactory introduction to the various types of transformations of $\mathbb{E}^n$ and $\mathbb{P}^n$; somehow the required material falls between the texts on Euclidean or projective geometry currently available.

Borsuk's book [45] is possibly the most comprehensive text for this purpose, but its notation is quite outdated.

If A is an $n \times n$ matrix, the inverse and transpose of A are denoted by A^{-1} and A^t. We call A *singular* or *nonsingular* according to whether $\det A = 0$ or $\det A \neq 0$, respectively; A^{-1} exists precisely when A is nonsingular. We also adopt the abbreviation A^{-t} for $(A^{-1})^t$. Note that if A is nonsingular, then A^t is also, and $(A^t)^{-1} = (A^{-1})^t$.

For transformations ϕ of $\mathbb{E}^n$ and $\mathbb{P}^n$, we shall permit ourselves the shorthand $\phi x = \phi(x)$. The reader may find Figure 0.1 useful in interpreting the definitions given below.

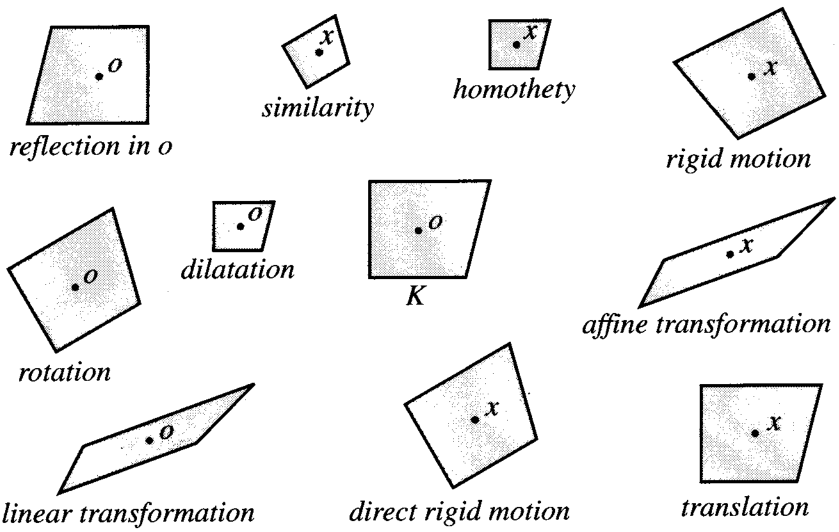

Figure 0.1. Transformations of a set K.

A *linear transformation* (or *affine transformation*) of $\mathbb{E}^n$ is a map ϕ from $\mathbb{E}^n$ to itself such that $\phi x = Ax$ (or $\phi x = Ax + t$, respectively), where A is an $n \times n$ matrix and $t \in \mathbb{E}^n$. (Here x is considered as a column vector, of course.) We call ϕ *singular* or *nonsingular* according to whether A is singular or nonsingular, respectively. The group of nonsingular linear (or affine) transformations is denoted by GL_n (or GA_n); its members are, in particular, bijections of $\mathbb{E}^n$ onto itself. The group of *special linear* (or *special affine*) transformations of $\mathbb{E}^n$ is denoted by SL_n (or SA_n, respectively). These are the members of GL_n (or GA_n) whose determinant is one. We shall write $\det \phi$ instead of $\det A$, and ϕ^{-1}, ϕ^t, and ϕ^{-t} for the affine transformations with corresponding matrices A^{-1}, A^t, and A^{-t}, respectively.

If A is the identity matrix, then $\phi x = x + t$, and the map ϕ is called a *translation*. Each affine transformation is composed of a linear transformation followed by a translation.

Any set of $n+1$ points in general position in $\mathbb{E}^n$ can be mapped onto any second set of $n+1$ points by a suitable affine transformation, and the latter is nonsingular if the second set is also in general position (see [368, Theorem 7, p. 16]).

Let ϕ be a linear transformation. If the vectors x and y are parallel, then $y = cx$ for real c and $\phi y = c\phi x$. It follows that ϕ takes parallel vectors onto parallel vectors. In fact (cf. [45, p. 156]), ϕ takes parallel k-dimensional planes onto parallel k-dimensional planes.

An *isometry* of $\mathbb{E}^n$ is a map ϕ such that $\|\phi x - \phi y\| = \|x - y\|$; in other words, a distance-preserving bijection. Isometries are also called congruences, and the image and pre-image under an isometry are said to be *congruent*. Every isometry is affine (see, for example, [45, p. 150] or [530, p. 139]). Examples of isometries are the translations and the *reflections*, which map all points to their mirror images in some fixed point, line, or plane. (In particular, $\phi x = -x$ is the reflection in the origin.)

If $F = S + x_0$ (where $S \in \mathcal{G}(n,k)$, $x_0 \in \mathbb{E}^n$, and $1 \le k \le n-1$) is a k-dimensional plane, and $x \in \mathbb{E}^n$, then there are unique points $y \in S$ and $z \in S^\perp$ such that $x = y + z$, and we can define a map taking x to $y + x_0 \in F$. This map is the (orthogonal) *projection* on the plane F. It is a singular affine transformation. If E is an arbitrary subset of $\mathbb{E}^n$, the image of E under a projection on a plane F is called the *projection of E on F* and denoted by $E|F$. Since $E|S$ is a translate of $E|F$ when $F = S + x_0$, we almost always work with the former.

If $\phi \in GL_n$, then

$$x \cdot \phi y = \phi^t x \cdot y, \tag{0.1}$$

for all $x, y \in \mathbb{E}^n$. The *orthogonal group O_n* of orthogonal transformations consists of those isometries of $\mathbb{E}^n$ that are also linear transformations; these are precisely the maps ϕ preserving the scalar product, that is, $\phi x \cdot \phi y = x \cdot y$. (An orthogonal matrix satisfies $A^t = A^{-1}$ and by (0.1) we have $\phi^t = \phi^{-1}$, hence the name.) It follows from this that orthogonal transformations have determinants with absolute value one. As is shown in [45, Theorem 50.6], every isometry is an orthogonal transformation followed by a translation, and for this reason isometries are sometimes also called *rigid motions*. The *special orthogonal group SO_n* of *rotations* about the origin consists of those orthogonal transformations with determinant one. A *direct rigid motion* is a rotation followed by a translation; these do not allow reflection.

A *dilatation* is a map $\phi x = rx$, for some $r > 0$. A *homothety* is a map $\phi x = rx + t$, for some $r > 0$ and $t \in \mathbb{E}^n$, that is, a composition of a dilatation with a translation (this is sometimes referred to as a direct homothety). A *similarity* is a composition of a dilatation with a rigid motion. We say two sets are *homothetic* (or *similar*) if one of them is an image of the other under a homothety (or similarity, respectively), or if one of the sets is a single point.

We find occasional use for projective transformations of $\mathbb{P}^n$. Such a transformation is given in terms of homogeneous coordinates by $\phi w = Aw + t$, where A is an $(n+1) \times (n+1)$ matrix and $t \in \mathbb{E}^{n+1}$, and where ϕ is called nonsingular if $\det A \neq 0$. Since we can regard $\mathbb{P}^n$ as $\mathbb{E}^n$ with a hyperplane H_∞ adjoined, we can also speak of a projective transformation of $\mathbb{E}^n$. In this regard, another formulation is useful. A *projective transformation* ϕ of $\mathbb{E}^n$ has the form

$$\phi x = \frac{\psi x}{x \cdot y + t}, \tag{0.2}$$

where $\psi \in GA_n$, $y \in \mathbb{E}^n$, and $t \in \mathbb{R}$, and ϕ is nonsingular if the associated linear map

$$\bar{\psi}(x, 1) = (\psi x, x \cdot y + t)$$

is nonsingular. If $y = o$, then ϕ is affine, but if $y \neq o$, ϕ maps the hyperplane $H = \{x : x \cdot y + t = 0\}$ onto H_∞. To avoid points in a set E being mapped into H_∞, we may insist that ϕ be *permissible* for E; this simply means that $E \cap H = \emptyset$.

Projective transformations map planes onto planes (neglecting the points mapping to or from infinity); see [368, pp. 19–20]. They also preserve cross ratio; a proof is given in [45, Corollary 96.11]. (The *cross ratio* of four points p_i, $1 \le i \le 4$, on a line is defined by

$$\langle p_1, \ldots, p_4 \rangle = \frac{\|p_3 - p_1\| \|p_4 - p_2\|}{\|p_4 - p_1\| \|p_3 - p_2\|}.)$$

Affine transformations are also projective transformations, so the former also preserve cross ratio.

The sets E and F are called *linearly, affinely*, or *projectively equivalent* if there is a nonsingular transformation ϕ, linear, affine, or projective and permissible for E, respectively, such that $\phi E = F$. Suppose that E and F are bounded centered sets affinely equivalent via a nonsingular transformation ϕ. If $\phi o = p$, then p is the center of F; but since o is the unique center of F, we have $p = o$. Therefore ϕ is linear, proving that E and F are linearly equivalent.

0.3. Basic convexity

There are several possibilities for an introduction to the basic properties of convex sets. For the absolute beginner, Lay's book [306] is recommended. The first chapter of [368], by McMullen and Shephard, is terse, but very informative, as is the first chapter of [459], by Schneider. The text of [535], by Yaglom and Boltyanskiĭ, is set out in the form of exercises and solutions, with plenty of helpful diagrams. Chapters 11 and 12 of Berger's two-volume set [23], [24], contain some wonderful pictures, and Lyusternik's little book [339] is quirky but delightful. A list of books on convexity can be found in [459, p. 433].

A set C in $\mathbb{E}^n$ is called *convex* if it contains the closed line segment joining any two of its points, or, equivalently, if $(1-t)x + ty \in C$ whenever $x, y \in C$ and $0 \le t \le 1$. A convex set, then, has no "holes" or "dents." A *convex body* is a compact convex set whose interior is nonempty; this definition conforms with general usage, but the reader is warned that in the important texts of Bonnesen and Fenchel [44] and Schneider [459] any compact convex set qualifies as a convex body. The *convex hull* conv E of a set E is the smallest convex set containing it.

If C is a compact convex set, a *diameter* of C is a chord $[x, y]$ of C such that $\|x - y\| = \operatorname{diam} C$.

A hyperplane H *supports* a set E at a point x if $x \in E \cap H$ and E is contained in one of the two closed half-spaces bounded by H. We say H is a *supporting hyperplane* of E if H supports E at some point.

A convex body is *strictly convex* if its boundary does not contain a line segment and *smooth* if there is a unique supporting hyperplane at each point of its boundary.

The intersection of a compact convex set with one of its supporting hyperplanes is called a *face*, and $(n-1)$-dimensional faces are also called *facets*. An *extreme point* of K is one not contained in the relative interior of any line segment contained in K. The point x is called an *exposed point* of K if there is a supporting hyperplane H such that $H \cap K = \{x\}$. Every exposed point is extreme, but the converse is not true. Also, a compact convex set is the closure of the convex hull of its exposed points, implying that every compact convex set has at least one exposed point (see [459, Section 1.4], especially Theorem 1.4.7). A *corner point* of a compact convex set in $\mathbb{E}^2$ is one at which there is more than one supporting line.

If K_1 and K_2 are disjoint compact convex sets in $\mathbb{E}^n$, then there is a hyperplane H that (strictly) *separates* K_1 and K_2; that is, K_1 is contained in one open half-space bounded by H, and K_2 in the other. A proof can be found in [306, Theorem 4.12] or [459, Theorem 1.3.7]. (In infinite-dimensional spaces, this separation theorem is closely related to the Hahn–Banach theorem; see [23, Section 11.4].)

Every affine transformation preserves convexity. If ϕ is a projective transformation, permissible for a line segment, then it maps this line segment onto another line segment. Therefore ϕ preserves the convexity of convex bodies for which it is permissible.

A nonempty subset C of $\mathbb{E}^n$ is a *cone* with vertex o if $ty \in C$ whenever $y \in C$ and $t \ge 0$. A *convex cone* with vertex o is a cone with vertex o that is convex; such a set is closed under nonnegative linear combinations. A cone (or convex cone) with vertex x is of the form $C + x$, where C is a cone (or convex cone, respectively) with vertex o.

Let us define some special convex bodies. The unit ball B in $\mathbb{E}^n$ was defined already. A *ball* is any set homothetic to B, and an *ellipsoid* is an affine image of B. The centered n-dimensional ellipsoids whose axes are parallel to the coordinate axes are of the form

$$\left\{x : \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} \leq 1\right\}.$$

If $0 \leq k \leq n$, a *k-dimensional simplex* in $\mathbb{E}^n$ is the convex hull of $k+1$ points in general position.

A *polyhedron* is a finite union of simplices; in $\mathbb{E}^2$, we shall use the term *polygon* instead. A convex polyhedron or *convex polytope* can also be defined as the convex hull of a finite set of points. We denote by $\mathcal{F}_k(P)$ the set of k-dimensional faces of a convex polytope P.

Important examples of convex polytopes are the *unit cube* $\{x : 0 \leq x_i \leq 1, 1 \leq i \leq n\}$ (and *centered unit cube* $\{x : |x_i| \leq 1/2, 1 \leq i \leq n\}$) in $\mathbb{E}^n$; the *parallelepipeds* or *parallelotopes*, affine images of the unit cube; the *boxes*, rectangular parallelepipeds with facets parallel to the coordinate hyperplanes; and the *cross-polytopes* (n-dimensional versions of the octahedron), each the convex hull of n mutually orthogonal line segments sharing the same midpoint. An n-dimensional *pyramid* P is the convex hull of an $(n-1)$-dimensional convex polytope Q (its *base*) and a point $x \notin \operatorname{aff} Q$ called the *apex* of P.

A (right spherical) *cylinder* in $\mathbb{E}^n$ is the Cartesian product of an $(n-1)$-dimensional ball C and a line segment orthogonal to aff C. A (right spherical) *bounded cone* in $\mathbb{E}^n$ is the convex hull of an $(n-1)$-dimensional ball C and a point on the line orthogonal to aff C through the center of C.

Topologically, a convex body is not very interesting. The surface of a convex body K in $\mathbb{E}^n$ is homeomorphic to S^{n-1} via a *radial map* f, defined by selecting a point $x_0 \in \operatorname{int} K$ and letting

$$f(x) = (x - x_0)/\|x - x_0\|, \tag{0.3}$$

for each $x \in \operatorname{bd} K$.

A real-valued function on $\mathbb{E}^n$ is *convex* if

$$f\big((1-t)x + ty\big) \leq (1-t)f(x) + tf(y),$$

for all $x, y \in \mathbb{E}^n$ and $0 \leq t \leq 1$, and *concave* if $-f$ is convex. (The terms concave up and concave down are sometimes used for convex and concave, respectively.)

0.4. The Hausdorff metric

Exactly what does it mean to say that a sequence of compact sets converges to another compact set? One must have a way of measuring the distance between two compact sets. This notion of distance must behave like the usual distance $d(x, y) = \|x - y\|$ between points, which has three fundamental properties: $d(x, y) \geq 0$, and equals zero if and only if $x = y$; $d(x, y) = d(y, x)$; and the triangle inequality

$$d(x, z) \leq d(x, y) + d(y, z).$$

Such a function is called a *metric*. We shall only define one metric for compact sets here, though there are several in common use (see Lemma 1.2.14 for another). The *Hausdorff metric* δ on the class of nonempty compact sets in $\mathbb{E}^n$ is defined by

$$\delta(E, F) = \max\{\max_{x\in E} d(x, F),\ \max_{x\in F} d(x, E)\}. \tag{0.4}$$

(A geometrically more appealing definition is given later.) It can be checked that δ satisfies the three conditions listed earlier. The proof, and basic properties of the metric space of compact sets in $\mathbb{E}^n$ defined in this way, may be found in [306, Section 14] or [459, Section 1.8]. For example, the space is complete, by [459, Theorem 1.8.2].

Suppose that E is a nonempty set in $\mathbb{E}^n$ and $\varepsilon > 0$. Then

$$E_\varepsilon = E + \varepsilon B = \cup_{x\in E}\,(x + \varepsilon B) \tag{0.5}$$

is called an *outer parallel set* of E. When E is closed, E_ε is just the set of all points whose distance from E is no more than ε. (See [306, Section 14], [459, p. 134]; see also the illustration in the book [495, Fig. 1.1(b)] of Stoyan, Kendall, and Mecke, and the interesting accompanying discussion on the utility of this idea in the processing of images.) This convenient concept allows the following alternative definition of the Hausdorff metric:

$$\delta(E, F) = \min\{\varepsilon > 0 :\ E \subset F_\varepsilon \text{ and } F \subset E_\varepsilon\}. \tag{0.6}$$

This means that the Hausdorff distance between two convex bodies K_1 and K_2 is at most ε if K_1 is contained in the outer parallel body $K_2 + \varepsilon B$ of K_2, and K_2 is contained in the outer parallel body $K_1 + \varepsilon B$ of K_1.

The Hausdorff metric is the standard one in the study of convex sets. We denote by $\mathcal{K}^n$ (or $\mathcal{K}_0^n$) the space of nonempty compact convex sets (or convex bodies, respectively) in $\mathbb{E}^n$ with the Hausdorff metric. (The definition of a body in Section 0.1 implies the existence of interior points when the set is nonempty.) It is the default metric, always used unless stated otherwise, for example, when discussing continuity of a function defined on the class of compact convex sets. A specific, and important, example of this is the continuity of volume on $\mathcal{K}^n$; see [306, Theorem 22.6] or [459, Theorem 1.8.16]. (One should try not to be blasé about such statements. After all, length is not continuous in $\mathbb{E}^2$, since one can approximate a closed line segment arbitrarily closely by polygonal arcs whose lengths are unbounded. According to Young [539, p. 303], this disturbed Lebesgue greatly when he was at school! In fact, length is only semicontinuous in $\mathbb{E}^2$.)

A very frequently quoted theorem is the following one, whose proof may be found in [306, Section 15] or [459, Theorem 1.8.6].

Theorem 0.4.1 (Blaschke's selection theorem). *Every bounded sequence of compact convex sets has a subsequence converging to a compact convex set.*

(A sequence of sets is *bounded* if there is a ball containing each member of the sequence.) In [459, Theorems 1.8.13 and 1.8.15], it is shown that each $K \in \mathcal{K}^n$ can be approximated arbitrarily closely from within or without by convex polytopes. This implies that the class of convex polytopes is dense in $\mathcal{K}^n$. It is also known that both the class of smooth convex bodies and the class of strictly convex bodies are dense in $\mathcal{K}^n$; see [459, Theorem 2.6.1].

0.5. Measure and integration

Measure theory deals with the definition and generalizations of the intuitive notions of length, area, and volume. The subject is amply supplied with well-written books appropriate for the novice. Many a student has learned the basics of Lebesgue measure and integration and the rudiments of general measure theory from [437], by Royden. At a slightly higher level, Munroe's book [397] is to be recommended. Unfortunately, however, the *geometric* aspects of measure theory are often ignored in the standard introductory texts. Exceptions are [530], by Weir (see Chapter 6 of Volume 1), and [258], by Jones (see Chapter 3). Of course, there are books on geometric measure theory proper, but here we can only suggest a browse of the first three of chapters of the entertaining and exquisitely illustrated introduction [396] by Morgan; we use no advanced geometric measure theory in this book.

In practice one can get by without most of the complicated theory of abstract measure. We summarize here the ingredients used in the sequel.

Consider, as a first example, area in the plane. Its essential properties are:

1. Familiar sets such as triangles, disks, and so on can be assigned a real number representing the area of the set.

2. The area of a countable union of disjoint sets is the sum of the areas of the sets; that is, area is *countably additive*.

3. The area of a set does not change when it is moved by a translation; that is, area is *translation invariant*. In fact, area is even invariant under isometries.

The same properties hold for a generalized notion of length in the real line, or volume in space. Length and area are denoted by λ_1 and λ_2, respectively. For Chapter 1, this is all one really needs.

Sooner or later, it becomes necessary to talk about the area of less familiar sets. It turns out that in order to retain the second and third properties, one has to give up the hope of assigning an area to *all* subsets of the plane (at least, if one wishes to use the commonly accepted axiom of choice). However, it can be shown that the concept of area can be defined so that all open sets can be assigned an area. Moreover, one can prove that the family of all sets that can be assigned an area forms a σ*-algebra*; that is, the family contains the empty set and is closed under the taking of complements and countable unions (and therefore also differences and countable intersections). Since the family of *Borel sets* is, by definition, the smallest σ-algebra containing the open sets, all Borel sets can be assigned an area.

Again, the same comments apply to generalized length in the real line and volume in space. Generalized length, area, and volume are examples of measures, and the sets that can be assigned a generalized length, area, or volume are called measurable sets. Among the measurable sets are those *of measure zero*, including all countable sets, but also many uncountable sets. For example, the Cantor ternary set in the real line has zero generalized length, and any line segment in the plane has zero area. Sets of measure zero (sometimes called *null sets*) are often neglected in measure theory, just as the number zero can be ignored in addition. For the types of measures encountered in this book, one is never too far from sanity when working with measurable sets, for it can be shown that each measurable set is the union of countably many closed sets and a (necessarily measurable) set of measure zero.

We are now ready for the formal definitions which abstract these ideas.

Let X be a set. A countably additive, extended real-valued function defined on a σ-algebra of subsets of X is called a *signed measure*; it is a *measure* if it is also nonnegative. The members of the σ-algebra are called *measurable sets*. We say a measure μ is *σ-finite* if X is a countable union of sets of finite μ-measure. A measure μ is said to be *concentrated* on a subset E of X if $\mu(X \setminus E) = 0$. If X is a topological space, and the σ-algebra consists of the Borel sets in X, the measure is called a *Borel measure*. An arbitrary measure in X is called *Borel regular* if Borel sets are measurable and every measurable set is contained in a Borel set of the same measure. A property is said to hold *μ-almost everywhere* or for *μ-almost all* $x \in X$ if there is a subset E of X with $\mu(E) = 0$ such that the property holds for all $x \in X \setminus E$.

We generally use lowercase Greek letters for measures. This is the convention adopted by most measure theorists, with the important exception of some who work in geometric measure theory, who use capital script letters, such as the $\mathcal{H}$ for Hausdorff measure (to be defined shortly). History has forced us to make, reluctantly, an exception for the area measures, defined in Section A.2.

After measures are defined, one can deal with the integral (some authors reverse this process). If μ is a measure in X, the *μ-measurable* extended real-valued functions are those for which the inverse image of an open set is a measurable set. When X is a topological space, there is also the class of *Borel functions* on X, the extended real-valued functions for which the inverse image of an open set is a Borel set. Every continuous function is Borel, and if μ is a Borel measure, then every Borel function is μ-measurable. For certain functions f on X, a meaning can be given to

$$\int_E f(x)\, d\mu(x),$$

the *integral* of f over the measurable set $E \subset X$, in such a way that in the familiar case of a nonnegative f defined on $\mathbb{E}^n$, the integral gives the volume under the graph of f. Nonnegative functions are called *μ-integrable* on E if they are μ-measurable and the integral exists and is finite. An arbitrary function f is *μ-integrable* if both its *positive part* f^+ and its *negative part* f^-, defined by

$$f^+(x) = \max\{f(x), 0\} \text{ and } f^-(x) = \max\{-f(x), 0\},$$

are integrable. A bounded measurable function is integrable on any set of finite measure. All this can be found in Chapters 4 and 11 of [437], for example.

One theorem in the theory of integration is of outstanding importance: Fubini's theorem (see [437, Theorem 19, p. 307]) says that in all reasonable circumstances, the integral of a function on a product of two spaces equals both of the two iterated integrals. (This allows, for example, the volume of a measurable set in $\mathbb{E}^3$ to be calculated by integrating the areas of its sections by planes parallel to a given plane.)

The *n-dimensional Lebesgue measure* λ_n in $\mathbb{E}^n$ is often defined to be the unique Borel-regular, translation-invariant measure in $\mathbb{E}^n$ such that the unit cube has unit measure. This provides one definition of generalized length in the real line, area in the plane, and volume in space. Defined this way, however, λ_n is not the most important measure. This honor goes to *k-dimensional Hausdorff measure* $\mathcal{H}^k$ in $\mathbb{E}^n$, $0 \le k \le n$. This is the standard way of measuring k-dimensional volume in $\mathbb{E}^n$, so that, for example, one could use $\mathcal{H}^1$ to measure the perimeter of a disc, or $\mathcal{H}^2$ for the surface area of a ball. The definition of Hausdorff measure (see the texts of Morgan [396, p. 8] or Rogers [433, Chapter 2]) is somewhat technical, but not really more so than the very commonly adopted definition of Lebesgue measure in the real line via Lebesgue outer measure, as in Chapter 3 of [437], for example.

It is a convenient fact that the two measures λ_n and $\mathcal{H}^n$ agree in $\mathbb{E}^n$ (see [396, Corollary 2.8] or [433, Theorem 30]), provided the correct constant is included in the definition of $\mathcal{H}^n$. There is a similar agreement between $\mathcal{H}^{n-1}$ and *n-dimensional spherical Lebesgue measure* in S^{n-1}, the unique Borel-regular, rotation-invariant measure in $\mathbb{E}^n$ such that S^{n-1} has measure equal to the constant ω_n whose value is given by (0.10). Indeed, it is well known that $\mathcal{H}^{n-1}$ is Borel regular and rotation invariant (see [433, Theorem 27 and p. 58]), and the fact that $\mathcal{H}^{n-1}(S^{n-1}) = \omega_n$ follows from integration via the area formula in [396, 3.7, p. 25]. Therefore we allow ourselves to speak loosely of k-dimensional Lebesgue measure in $\mathbb{E}^n$ when we really mean k-dimensional Hausdorff measure, and use λ_k for integration in planes or spheres. Two abbreviations should be noted: We shall write dx for $d\lambda_n(x)$ in $\mathbb{E}^n$ and du for $d\lambda_{n-1}(u)$ in S^{n-1}.

The measure $\mathcal{H}^0$ (we shall write λ_0) is just the counting measure, which counts the number of points in a set.

When no misunderstanding can arise – for example, when working with compact convex sets – we sometimes call the λ_k-measure of a k-dimensional body in $\mathbb{E}^n$ its *volume*. This is traditional in geometry.

Often we want to work with the equivalence classes of measurable sets modulo sets of measure zero, and here it is useful to write $E \simeq F$ when $\lambda_n(E \triangle F) = 0$.

Let $\phi \in GA_n$. Then $|\det \phi|$ is the factor by which ϕ changes volume, that is,

$$\lambda_n(\phi E) = |\det \phi| \lambda_n(E), \tag{0.7}$$

for each λ_n-measurable set E in $\mathbb{E}^n$; see [530, pp. 142–4]. It follows that the members of SA_n, and more generally those maps in GA_n whose determinants are ± 1, are volume preserving. It also follows that if $r \geq 0$, then $\lambda_n(rE) = r^n \lambda_n(E)$. More generally, if $1 \leq k \leq n$, E is a λ_k-measurable set in $\mathbb{E}^n$, and $r \geq 0$, then

$$\lambda_k(rE) = r^k \lambda_k(E). \tag{0.8}$$

We saw in Section 0.2 that ϕ takes parallel k-dimensional planes onto parallel k-dimensional planes. If $x - x' = y - y'$, then $\phi x - \phi x' = \phi y - \phi y'$, so ϕ preserves the equality of lengths of parallel vectors. More generally, ϕ preserves the ratio of λ_k-measures of sets in parallel k-dimensional planes.

The volume of the unit ball in $\mathbb{E}^n$ is given by

$$\kappa_n = \lambda_n(B) = \frac{\pi^{n/2}}{\Gamma(1 + \frac{n}{2})}, \tag{0.9}$$

with the convention $\kappa_0 = 1$, and its surface area is

$$\omega_n = \lambda_{n-1}(S^{n-1}) = n\kappa_n. \tag{0.10}$$

The first computation is given in [348, pp. 324–5] and the second in [96, p. 125]; or see [463, p. 18]. To calculate special values of κ_n, one only needs $\Gamma(1 + x) = x\Gamma(x)$, $\Gamma(1) = 1$, and $\Gamma(1/2) = \sqrt{\pi}$. It is interesting that κ_n increases with n to its maximum value $8\pi^2/15$ when $n = 5$, and then decreases, approaching zero.

Using (0.9) and (0.7), one shows that the n-dimensional centered ellipsoid $\{x : \sum_{i=1}^n x_i^2/a_i^2 \leq 1\}$ has volume

$$a_1 a_2 \cdots a_n \kappa_n. \tag{0.11}$$

The volume of a parallelepiped is the λ_{n-1}-measure of its base times its height orthogonal to its base. The volume of the parallelepiped in $\mathbb{E}^n$ with vertices at o, $p_1, \ldots, p_n$ is also given by

$$|\det(p_{ij})|, \tag{0.12}$$

where $p_i = (p_{i1}, \ldots, p_{in})$, and the volume of the simplex in $\mathbb{E}^n$ with vertices at $o, p_1, \ldots, p_n$ is

$$\frac{1}{n!}|\det(p_{ij})|, \tag{0.13}$$

as in [45, p. 117]. We have the formula

$$\lambda_n(P) = \frac{1}{n} z\lambda_{n-1}(Q) \tag{0.14}$$

for the volume of a pyramid or bounded cone P with base Q and height (the distance from aff Q to the apex) z. This is easily obtained by integration and induction, as in [23, 9.12.4.4] for the simplex; Dehn's solution of Hilbert's third problem indicates that some form of limit argument is required (see the discussion in [24, 12.2.5.2], for example).

We occasionally need other Borel measures in $\mathbb{E}^n$ or S^{n-1}. A signed Borel measure μ in S^{n-1} is called *even* (or *odd*) if $\mu(-E) = \mu(E)$ (or $\mu(-E) = -\mu(E)$, respectively), for all Borel sets E.

Let μ be a measure in $\mathbb{E}^n$ and E a bounded set in $\mathbb{E}^n$ of finite positive μ-measure. The *centroid* of E with respect to μ is the point

$$c = \frac{1}{\mu(E)} \int_E x \, d\mu(x). \tag{0.15}$$

The centroid of E is contained in conv E; see [44, Section 6, p. 9].

There is another measure that is extremely important in geometry, and it occurs in this fashion. It is sometimes essential to be able to measure the size of a set of lines or planes, or to integrate a function defined on a set of lines or planes. We only need to do this for sets of subspaces, that is, lines and planes containing the origin, or generally for subsets of $\mathcal{G}(n, k)$. Moreover, our measure should be compatible with the appropriate geometric transformations, so that, for example, the measure of a subset E of $\mathcal{G}(n, k)$ should equal the measure of the set obtained by applying the same rotation about the origin to each member of E. For $k = 1$ (or $k = n - 1$), this is easy: Just identify each 1-dimensional subspace (or $(n-1)$-dimensional subspace) S with the corresponding antipodal pair of points $\pm u$ in S^{n-1} such that the vector u is parallel to S (or orthogonal to S, respectively), and then use the measure λ_{n-1} in S^{n-1}. For $1 < k < n - 1$, however, one needs a new measure, which can be defined by the following general process.

Let X be a locally compact topological group. Then there is a nonzero Borel-regular measure μ in X that is also invariant under left translations by elements of X. This measure μ is called the *Haar measure* in X; it is unique up to multiplication by a constant, and is finite if X is compact. A detailed proof of its existence and uniqueness is given in the texts of Cohn [94, Chapter 9] and Munroe [397, Section 17], for example. However, for the special case of most interest here, this can be avoided. A clever direct construction due to Schneider